18.06 Final Exam May 18, 2010 Professor Strang

1. **(12 points)** This question is about the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}.$$

(a) Find a lower triangular L and an upper triangular U so that $A = LU$.

Answer:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the reduced row echelon form $R = rref(A)$. How many independent columns in A?

Answer: 2

$$R = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \text{ in this example.}$$

(c) Find a basis for the nullspace of A.

Answer:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(d) If the vector b is the sum of the four columns of A, write down the complete solution to $Ax = b$.

Answer:

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

2. **(11 points)** This problem finds the curve $y = C + D\,2^t$ which gives the best least squares fit to the points $(t, y) = (0, 6), (1, 4), (2, 0)$.

(a) Write down the 3 equations that would be satisfied *if* the curve went through all 3 points.

Answer:

$$\begin{aligned} \text{C} + \text{1D} &= 6 \\ \text{C} + \text{2D} &= 4 \\ \text{C} + \text{4D} &= 0 \end{aligned}$$

(b) Find the coefficients C and D of the best curve $y = C + D2^t$.

Answer:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

Solve $A^T A\hat{x} = A^T b$:

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix} \text{gives} \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.$$

(c) What values should y have at times $t = 0, 1, 2$ so that the best curve is $y = 0$?

Answer:

The projection is $p = (0, 0, 0)$ if $A^T b = 0$. In this case, $b =$ values of $y = c(2, -3, 1)$.

3. **(11 points)** Suppose $Av_i = b_i$ for the vectors $v_1, \ldots, v_n$ and $b_1, \ldots, b_n$ in R^n. Put the v's into the columns of V and put the b's into the columns of B.

(a) Write those equations $Av_i = b_i$ in matrix form. *What condition on which vectors* allows A to be determined uniquely? Assuming this condition, *find A from V and B.*

Answer:

$A\,[v_1 \cdots v_n] = [b_1 \cdots b_n]$ or $AV = B$. Then $A = BV^{-1}$ if the $v's$ are independent.

(b) Describe the column space of that matrix A in terms of the given vectors.

Answer:

The column space of A consists of all linear combinations of $b_1, \cdots, b_n$.

(c) What additional condition on which vectors makes A an *invertible* matrix? Assuming this, find A^{-1} from V and B.

Answer:

If the $b's$ are independent, then B is invertible and $A^{-1} = VB^{-1}$.

4. **(11 points)**

(a) Suppose x_k is the fraction of MIT students who prefer calculus to linear algebra at year k. The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k+1$, $1/5$ of those who prefer calculus change their mind (possibly after taking 18.03). Also at year $k+1$, $1/10$ of those who prefer linear algebra change their mind (possibly because of this exam).

Create the matrix A to give $\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}$ and find the limit of $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as $k \to \infty$.

Answer:

$$A = \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}.$$

The eigenvector with $\lambda = 1$ is $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

This is the steady state starting from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\frac{2}{3}$ of all students prefer linear algebra! I agree.

(b) Solve these differential equations, starting from $x(0) = 1, \quad y(0) = 0$:

$$\frac{dx}{dt} = 3x - 4y \qquad \frac{dy}{dt} = 2x - 3y\,.$$

Answer:

$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}.$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $x_1 = (2,1)$ and $x_2 = (1,1)$.
The initial vector $(x(0), y(0)) = (1,0)$ is $x_1 - x_2$.
So the solution is $(x(t), y(t)) = e^t(2,1) + e^{-t}(1,1)$.

(c) For what initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$ does the solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ to this differential equation lie on a single straight line in R^2 for all t?

Answer:

If the initial conditions are a multiple of either eigenvector $(2, 1)$ or $(1, 1)$, the solution is at all times a multiple of that eigenvector.

5. **(11 points)**

(a) Consider a 120° rotation around the axis $x = y = z$. Show that the vector $i = (1, 0, 0)$ is rotated to the vector $j = (0, 1, 0)$. (Similarly j is rotated to $k = (0, 0, 1)$ and k is rotated to i.) How is $j - i$ related to the vector $(1, 1, 1)$ along the axis?

Answer:

$$j - i = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

is orthogonal to the axis vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So are $k - j$ and $i - k$. By symmetry the rotation takes i to j, j to k, k to i.

(b) Find the matrix A that produces this rotation (so Av is the rotation of v). Explain why $A^3 = I$. What are the eigenvalues of A?

Answer:

$A^3 = I$ because this is three 120° rotations (so 360°). The eigenvalues satisfy $\lambda^3 = 1$ so $\lambda = 1, e^{2\pi i/3}, e^{-2\pi i/3} = e^{4\pi i/3}$.

(c) If a 3 by 3 matrix P projects every vector onto the plane $x+2y+z = 0$, find three eigenvalues and three independent eigenvectors of P. No need to compute P.

Answer: The plane is perpendicular to the vector $(1, 2, 1)$. This is an eigenvector of P with $\lambda = 0$. The vectors $(-2, 1, 0)$ and $(1, -1, 1)$ are eigenvectors with $\lambda = 0$.

6. **(11 points)** This problem is about the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}.$$

(a) Find the eigenvalues of A^TA and also of AA^T. For both matrices find a complete set of orthonormal eigenvectors.

Answer:

$$A^TA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 28 \\ 28 & 56 \end{bmatrix}$$

has $\lambda_1 = 70$ and $\lambda_2 = 0$ with eigenvectors $x_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\mathrm{AA}^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 15 & 30 & 45 \end{bmatrix}$ has $\lambda_1 = 70$, $\lambda_2 = 0$, $\lambda_3 = 0$ with

$\mathrm{x}_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}$.

(b) If you apply the Gram-Schmidt process (orthonormalization) to the columns of this matrix A, what is the resulting output?

Answer:

Gram-Schmidt will find the unit vector

$$q_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

But the construction of q_2 fails because column $2 = 2$ (column 1).

(c) If A is *any* m by n matrix with $m > n$, tell me why AA^T cannot be positive definite. Is A^TA always positive definite? (If not, what is the test on A?)

Answer

AA^T is m by m but its rank is not greater than n (all columns of AA^T are combinations of columns of A). Since $n < m$, AA^T is singular.
A^TA is positive definite if A has full colum rank n. (Not always true, A can even be a zero matrix.)

7. **(11 points)** This problem is to find the determinants of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(a) Find $\det A$ and give a reason.

Answer:

$\det A = 0$ because two rows are equal.

(b) Find the cofactor C_{11} and then find $\det B$. This is the volume of what region in R^4?

Answer:

The cofactor $C_{11} = -1$. Then $\det B = \det A - C_{11} = 1$. This is the volume of a box in R^4 with edges $=$ rows of B.

(c) Find $\det C$ for any value of x. You could use linearity in row 1.

Answer:

$\det C = xC_{11} + \det B = -x + 1$. Check this answer (zero), for $x = 1$ when $C = A$.

8. **(11 points)**

(a) When A is similar to $B = M^{-1}AM$, prove this statement:

If $A^k \to 0$ when $k \to \infty$, then also $B^k \to 0$.

Answer:

A and B have the same eigenvalues. If $A^k \to 0$ then all $|\lambda| < 1$. Therefore $B^k \to 0$.

(b) Suppose S is a fixed invertible 3 by 3 matrix.

This question is about all the matrices A that are diagonalized by S, so that $S^{-1}AS$ is diagonal. Show that these matrices A form a subspace of 3 by 3 matrix space. (Test the requirements for a subspace.)

Answer:

If A_1 and A_2 are in the space, they are diagonalized by S. Then $S^{-1}(cA_1 + dA_2)S$ is diagonal + diagonal = diagonal.

(c) Give a basis for the space of 3 by 3 *diagonal matrices*. Find a basis for the space in part (b) — all the matrices A that are diagonalized by S.

Answer:

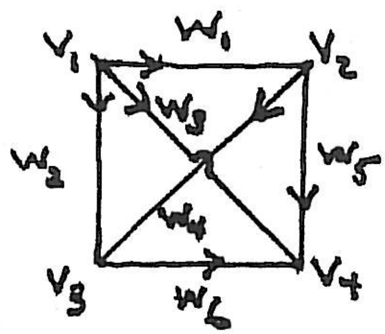
A basis for the diagonal matrices is

$$D_1 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} D_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} D_3 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

Then $SD_1S^{-1}, SD_2S^{-1}, SD_3S^{-1}$ are all diagonalized by S: a basis for the subspace.

9. **(11 points)** This square network has 4 nodes and 6 edges. On each edge, the direction of positive current $w_i > 0$ is from lower node number to higher node number. The voltages at the nodes are $(v_1, v_2, v_3, v_4.)$

Answer:



(a) Write down the incidence matrix A for this network (so that Av gives the 6 voltage differences like $v_2 - v_1$ across the 6 edges). What is the rank of A? What is the dimension of the nullspace of A^T?

Answer:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

has rank $r = 3$. The nullspace of A^T has dimension $6 - 3 = 3$.

(b) Compute the matrix $A^T A$. What is its rank? What is its nullspace?

Answer:

$$A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

has rank 3 like A. The nullspace is the line through $(1, 1, 1, 1)$.

(c) Suppose $v_1 = 1$ and $v_4 = 0$. If each edge contains a unit resistor, the currents $(w_1, w_2, w_3, w_4, w_5, w_6)$ on the 6 edges will be $w = -Av$ by Ohm's Law. Then Kirchhoff's Current Law (flow in = flow out at every node) gives $A^T w = 0$ which means $\boldsymbol{A^T A v = 0}$. Solve $A^T A v = 0$ for the unknown voltages v_2 and v_3. Find all 6 currents w_1 to w_6. How much current enters node 4?

Answer:

Note: As stated there is no solution (my apologies!). All solutions to $A^T A v = 0$ are multiples of $(1, 1, 1, 1)$ which rules out $v_1 = 1$ and $v_4 = 0$.
Intended problem: I meant to solve the reduced equations using KCL only at nodes 2 and 3. In fact symmetry gives $v_2 = v_3 = \frac{1}{2}$. Then the currents are $w_1 = w_2 = w_5 = w_6 = \frac{1}{2}$ around the sides and $w_3 = 1$ and $w_4 = 0$ (symmetry). So $w_3 + w_5 + w_6 = \frac{1}{2}$ is the total current into node 4.