18.06 PSET 3 SOLUTIONS

FEBRUARY 22, 2010

Problem 1. (§3.2, #18) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \\ 0 \\ 1 \end{bmatrix}.$$

Solution. **(4 points)** The equation $x = 12 + 3y + z$ says it all:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \left(= \begin{bmatrix} 12 + 3y + z \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} \boxed{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \boxed{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \boxed{1} \\ 0 \\ 1 \end{bmatrix}.$$

□

Problem 2. (§3.2, #24) (If possible...) Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.

Solution. **(4 points)** $\boxed{\text{Not possible}}$: Such a matrix A must be 3×3. Since the nullspace is supposed to contain two independent vectors, A can have at most $3 - 2 = 1$ pivots. Since the column space is supposed to contain two independent vectors, A must have at least 2 pivots. These conditions cannot both be met! □

Problem 3. (§3.2, #36) How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

Solution. **(12 points)** $\boxed{\mathbf{N}(C) = \mathbf{N}(A) \cap \mathbf{N}(B)}$ just the intersection: Indeed,

$$C\mathbf{x} = \begin{bmatrix} A\mathbf{x} \\ B\mathbf{x} \end{bmatrix}$$

so that $C\mathbf{x} = 0$ if and only if $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$. (...and as a nitpick, it wouldn't be quite sloppy instead write "if and only if $A\mathbf{x} = B\mathbf{x} = 0$"—those are zero vectors of potentially different length, hardly equal). □

Problem 4. (§3.2, #37) Kirchoff's Law says that *current in = current out* at every node. This network has six currents $y_1, \dots, y_6$ (the arrows show the positive direction, each y_i could be positive or negative). Find the four equations $A\mathbf{y} = 0$ for Kirchoff's Law at the four nodes. Find three special solutions in the nullspace of A.

Solution. **(12 points)** The four equations are, in order by node,

$$\begin{aligned} y_1 - y_3 + y_4 &= 0 \\ -y_1 + y_2 + y_5 &= 0 \\ -y_2 + y_3 + y_6 &= 0 \\ -y_4 - y_5 - y_6 &= 0 \end{aligned}$$

or in matrix form $A\mathbf{y} = 0$ for

$$\boxed{A = \begin{bmatrix} 1 & 0-1 & 1 & 0 & 0 & \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}}$$

Adding the last three rows to the first eliminates it, and shows that we have three "pivot variables" y_1, y_2, y_4 and three "free variables" y_3, y_5, y_6. We find the special solutions by back-substitution from $(y_3, y_5, y_6) = (1,0,0), (0,1,0), (0,0,1)$:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$ □

Problem 5. (§3.3, #19) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA = I$ (which is not so obvious!).

Solution. **(4 points)** Since A is n by n, $\operatorname{rank}(A) \leq n$ and conversely

$$n = \operatorname{rank}(I_n) = \operatorname{rank}(AB) \leq \operatorname{rank}(A).$$

The rest of the problem statement seems to be "commentary," and not further things to do. □

Problem 6. (§3.3, #25) *Neat fact* **Every m by n matrix of rank r reduces to (m by r) times (r by n):**

$$A = (\text{pivot columns of } A)\,(\text{first } r \text{ rows of } R)) = (\mathbf{COL})\,(\mathbf{ROW})\,.$$

Write the 3 by 4 matrix A in equation (1) at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 4 matrix from R.

Solution. **(4 points)**

$$\boxed{A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{bmatrix}}$$ □

Problem 7. (§3.3, #27) Suppose R is m by n of rank r, with pivot columns first:

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

(a) What are the shapes of those four blocks?
(b) Find a *right-inverse* B with $RB = I$ if $r = m$.
(c) Find a *left-inverse* C with $CR = I$ if $r = n$.
(d) What is the reduced row echelon form of R^T (with shapes)?
(e) What is the reduced row echelon form of $R^T R$ (with shapes)?

Prove that $R^T R$ has the same nullspace as R. Later we show that $A^T A$ always has the same nullspace as A (a valuable fact).

Solution. **(12 points)**

(a)

$$\boxed{\begin{bmatrix} r \times r & r \times (n-r) \\ (m-r) \times r & (m-r) \times (n-r) \end{bmatrix}}$$

(b) In this case

$$R = \begin{bmatrix} I & F \end{bmatrix} \qquad \text{so we can take} \qquad \boxed{B = \begin{bmatrix} I_{r\times r} \\ 0_{(n-r)\times r} \end{bmatrix}}$$

(c) In this case

$$R = \begin{bmatrix} I & 0 \end{bmatrix} \qquad \text{so we can take} \qquad \boxed{C = \begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \end{bmatrix}}$$

(d) Note that

$$R^T = \begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \\ F^T & 0_{(n-r)\times(m-r)} \end{bmatrix} \qquad \text{so that} \qquad rref(R^T) = \boxed{\begin{bmatrix} I_{r\times r} & 0_{r\times(m-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(m-r)} \end{bmatrix}}$$

(e) Note that

$$R^T R = \begin{bmatrix} I_{r\times r} & F \\ F^T & 0 \end{bmatrix} \qquad \text{so that} \qquad rref(R^T R) = \boxed{\begin{bmatrix} I_{r\times r} & F_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}} = R$$

Performing row operations doesn't change the nullspace, so that $\mathbf{N}(A) = \mathbf{N}(rref(A))$ for any matrix A. So, $\mathbf{N}(A) = \mathbf{N}(R^T R)$ by (e). □

Problem 8. (§3.3, #28) Suppose you allow elementary *column* operations on A as well as elementary row operations (which get to R). What is the "row-and-column reduced form" for an m by n matrix of rank r?

Solution. **(12 points)** After getting to R we can use the column operations to get rid of F, and get to

$$\boxed{\begin{pmatrix} I_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}}$$

□

Problem 9. (§3.3, #17 – Optional)

(a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columnd of AB. Then AB cannot have new pivot columns, so $\text{rank}(AB) \le \text{rank}(B)$.

(b) Find A_1 and A_2 so that $\text{rank}(A_1 B) = 1$ and $\text{rank}(A_2 B) = 0$ for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution. **(Optional)**

(a) That column j of B is a combination of previous columns of B means precisely that there exist numbers $a_1, \ldots, a_{j-1}$ so that each row vector $\mathbf{x} = (x_i)$ of B satisfies the linear relation

$$x_j = \sum_{i=1}^{j-1} a_i x_i = a_1 x_1 + \cdots + a_{j-1} x_{j-1}$$

The rows of the matrix AB are all linear combinations of the rows of B, and so also satisfy this linear relation. So, column j is the same combination of previous columns of AB, as desired. Since a column is pivot column precisely when it is not a combination of previous columns, this shows that AB cannot have previous columns and the rank inequality.

(b) Take $A_1 = I_2$ and $A_2 = 0_2$ (or for a less trivial example $A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$). □

Problem 10. (§3.4, #13) Explain why these are all false:

(a) The complete solution is any linear combination of $\mathbf{x}_p$ and $\mathbf{x}_n$.

(b) A system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.

(c) The solution $\mathbf{x}_p$ with all free variables zero is the shortest solution (minimum length $\|x\|$). Find a 2 by 2 counterexample.

(d) If A is invertible there is no solution $\mathbf{x}_n$ in the nullspace.

Solution. **(4 points)**

(a) The coefficient of $\mathbf{x}_p$ must be one.

(b) If $\mathbf{x}_n \in \mathbf{N}(A)$ is in the nullspace of A and $\mathbf{x}_p$ is one particular solution, then $\mathbf{x}_p + \mathbf{x}_n$ is also a particular solution.

(c) Lots of counterexamples are possible. Let's talk about the 2 by 2 case geometrically: If A is a 2 by 2 matrix of rank 1, then the solutions to $A\mathbf{x} = \mathbf{b}$ form a line parallel to the line that is the nullspace. We're asking that this line's closest point to the origin be somewhere not along an axis. The line $x + y = 1$ gives such an example.

Explicitly, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_p = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Then, $\|\mathbf{x}_p\| = 1/\sqrt{2} < 1$ while the particular solutions having some coordinate equal to zero are $(1, 0)$ and $(0, 1)$ and they both have $\| \cdot \| = 1$.

(d) There's always $\mathbf{x}_n = 0$. $\square$

Problem 11. (§3.4, #25) Write down all known relations between r and m and n if $A\mathbf{x} = \mathbf{b}$ has

(a) no solution for some $\mathbf{b}$

(b) infinitely many solutions for every $\mathbf{b}$

(c) exactly one solution for some $\mathbf{b}$, no solution for other $\mathbf{b}$

(d) exactly one solution for every $\mathbf{b}$.

Solution. **(4 points)**

(a) The system has less than full row rank: $r < m$.

(b) The system has full row rank, and less than full column rank: $m = r < n$.

(c) The system has full column rank, and less than full row rank: $n = r < m$.

(d) The system has full row and column rank (i.e., is invertible): $n = r = m$. $\square$

Problem 12. (§3.4, #28) Apply Gauss-Jordan elimination to $U\mathbf{x} = 0$ and $U\mathbf{x} = \mathbf{c}$. Reach $R\mathbf{x} = 0$ and $R\mathbf{x} = \mathbf{d}$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} U & \mathbf{c} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix}.$$

Solve $R\mathbf{x} = 0$ to find $\mathbf{x}_n$ (its free variable is $x_2 = 1$). Solve $R\mathbf{x} = \mathbf{d}$ to find $\mathbf{x}_p$ (its free variable is $x_2 = 0$).

Solution. **(4 points)** Let me just say to whoever's reading: The problem statement is confusing as written!! In any case, I *think* the desired response is:

$$\begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix} \qquad \text{and} \begin{bmatrix} 1 & 2 & 0 & -\mathbf{1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$$

so that

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{d} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and

$$\mathbf{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

$\square$

Problem 13. (§3.4, #35) Suppose K is the 9 by 9 second difference matrix (2's on the diagonal, -1's on the diagonal above and also below). Solve the equation $K\mathbf{x} = \mathbf{b} = (10, \dots, 10)$. If you graph $x_1, \dots, x_9$ above the points $1, \dots, 9$ on the x axis, I think the nine points fall on a parabola.

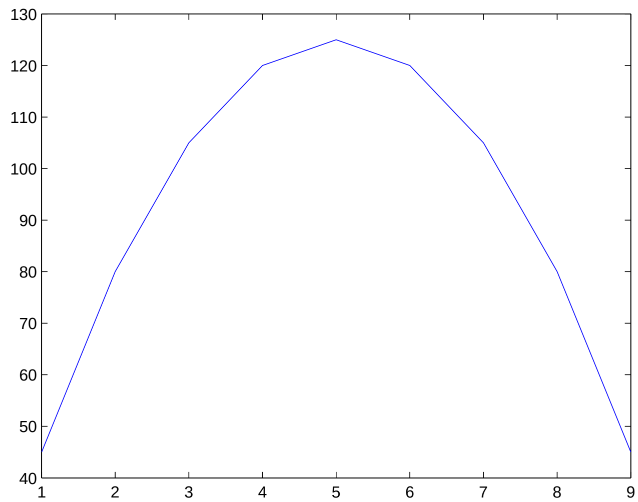
Solution. **(12 points)** Here's some MATLAB code that should do this:

```
K = 2*eye(9)  +  diag(-1*ones(1,8),1)  +  diag(-1*ones(1,8),-1);
b = 10*ones(9,1);
x = K \ b
```

It gives back that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \boxed{\begin{bmatrix} 45 \\ 80 \\ 105 \\ 120 \\ 125 \\ 120 \\ 105 \\ 80 \\ 45 \end{bmatrix}}$$

And for fun, the graph is indeed parabola-like:



$\square$

Problem 14. (§3.4, #36) Suppose $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{b}$ have the same (complete) solutions for every $\mathbf{b}$. Is it true that $A = C$?

Solution. **(12 points)** $\boxed{\text{Yes}}$. In order to check that $A = C$ as matrices, it's enough to check that $A\mathbf{y} = C\mathbf{y}$ for all vectors $\mathbf{y}$ of the correct size (or just for the standard basis vectors, since multiplication by them "picks out the columns"). So let $\mathbf{y}$ be any vector of the correct size, and set $\mathbf{b} = A\mathbf{y}$. Then $\mathbf{y}$ is certainly a solution to $A\mathbf{x} = \mathbf{b}$, and so by our hypothesis must also be a solution to $C\mathbf{x} = \mathbf{b}$; in other words, $C\mathbf{y} = \mathbf{b} = A\mathbf{y}$. $\square$